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Synchronization and resonance in a driven system of coupled oscillators

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We study the noise effects in a driven system of globally coupled oscillators, with particular attention to the interplay between driving and noise. The self-consistency equation for the order parameter, which measures the collective synchronization of the system, is derived; it is found that the total order parameter decreases monotonically with noise, indicating overall suppression of synchronization. Still, for large coupling strengths, there exists an optimal noise level at which the periodic (ac) component of the order parameter reaches its maximum. The response of the phase velocity is also examined and found to display resonance behavior. [S1063-651X(99)14310-1]

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I. INTRODUCTION

The set of coupled nonlinear oscillators serves as a prototype model for a variety of self-organizing systems in physics and in other sciences, which display the remarkable phenomena of collective synchronization [1–4]. Due to analytic simplicity and some physical as well as biological applications, the system with global coupling has been mostly studied both analytically and numerically [5–9]. Here external periodic driving may induce characteristic mode locking of each oscillator, leading the system to display periodic synchronization [10]. In such a driven system, the presence of noise raises another interesting possibility of *stochastic resonance* (SR), which leads to the amplification of the response of the system by cooperative interactions between the noise and external periodic driving [11]. The SR phenomena, which have various practical applications [12–17], have been investigated in systems with relatively few degrees of freedom, and observed in bistable systems and also in systems with periodic potentials [14]. On the other hand, the SR effects have hardly been examined in a system with many degrees of freedom such as the system of coupled oscillators [18].

In this paper we consider a system of globally coupled stochastic oscillators, driven periodically, and investigate the interplay of noise and periodic driving, with particular attention to the possibility of stochastic resonance. For this purpose, it is crucial to consider appropriate responses of the system to the periodic forcing. Here we consider the response of the phase velocity, as well as the order parameter, which describes the phase synchronization. We first derive the self-consistency equation for the order parameter and investigate the behavior of the order parameter in the presence of noise. It is found that the total order parameter, which consists of the time-independent (dc) and periodic (ac) components, decreases monotonically with noise, indicating the overall suppression of phase synchronization. The ac component, on the other hand, may first increase as noise grows from zero, and reach its maximum at a finite noise level. Such SR-like behavior is also observed in the response of the phase velocity; at low noise levels, the noise subtracted power spectrum of the phase velocity tends to increase with noise.

This paper consists of six sections. Section II introduces the driven system of coupled oscillators subject to random noise. The recurrence relation for the Fourier components is obtained. In Sec. III, we use the recurrence relation obtained in Sec. II, and derive the self-consistency equation for the order parameter. The set of coupled equations of motion for the system is transformed into a Fokker-Planck equation and the corresponding probability density is expanded as a Fourier series. Sections IV and V are devoted to the investigation of the responses of the phase and of the phase velocity, respectively, to the external driving. In spite of the overall suppression of synchronization, the ac component of the order parameter, corresponding to the phase response, as well as the response of the phase velocity is revealed to display SR-like behavior. Finally, a brief summary is given in Sec. VI.

II. DRIVEN SYSTEM OF COUPLED OSCILLATORS

The set of equations of motion governing the dynamics of the system of N coupled oscillators is given by

$$\dot{\phi}_i+\frac{K}{N}\sum_{j=1}^{N}\sin(\phi_i-\phi_j)=\omega_i+I_i\cos\Omega t+\Gamma_i(t)\quad(i=1,2,\ldots,N),\quad(1)$$

where ϕ_i represents the phase of the ith oscillator. The second term on the left-hand side corresponds to the global coupling between oscillators, with strength K/N. The first and the second terms on the right-hand side describe the natural frequency of the ith oscillator and the periodic driv-

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ing on the ith oscillator, respectively. Finally, $\Gamma_i(t)$ is independent white noise with zero mean and correlation,

$$\langle \Gamma_i(t)\Gamma_j(t')\rangle = 2D\,\delta_{ij}\,\delta(t-t'), \tag{2}$$

where $D(>0)$ plays the role of the ''effective temperature'' of the system. The natural frequency ω_i is distributed over the whole oscillators according to the distribution $g(\omega)$, which is assumed to be smooth and symmetric about ω_0. Without loss of generality, we may take ω_0 to be zero and assume that $g(\omega)$ is concave at $\omega=0$, i.e., $g''(0)<0$. The periodic (ac) driving amplitude I_i may also vary for different oscillators, while the frequency Ω of the driving is assumed to be uniform for all oscillators. In the absence of noise ($D=0$), Eq. (1) precisely reduces to the set of equations of motion studied in Ref. [10]. The set of equations of motion in Eq. (1) describes a superconducting wire network [19] and may also be regarded as the mean-field version of an array of resistively shunted junctions, which serves as a common model for describing the dynamics of superconducting arrays [20]. In these cases, the two terms on the right-hand side of Eq. (1) correspond to the combined direct and alternating current bias.

Collective behavior of such an N-oscillator system is conveniently described by the complex *order parameter*

$$\Psi \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j} = \Delta e^{i\theta}, \tag{3}$$

where nonvanishing Ψ indicates emergence of synchronization. Note that the synchronized state corresponds to the superconducting state with global phase coherence in the case of the superconducting network or array. The order parameter defined in Eq. (3) allows us to reduce Eq. (1) into a *single* decoupled equation

$$\dot{\phi}_i + K\Delta\,\sin(\phi_i-\theta) = \omega_i + I_i\cos\Omega t + \Gamma_i(t).$$

We then seek the stationary solution with θ being constant, which is possible due to the symmetry of the distribution of ω_i and I_i about zero. Redefining $\phi_i-\theta$ as ϕ_i and suppressing indices, we obtain the reduced equation of motion

$$\dot{\phi} + K\Delta\,\sin\phi = \omega + I\cos\Omega t + \Gamma(t), \tag{4}$$

which depends explicitly on the order parameter. In this manner the order parameter Δ, defined in terms of the phase via Eq. (3), in turn determines the behavior of the phase via Eq. (4), and can thus be obtained by imposing self-consistency, as discussed in Sec. III. Note here that Δ, in general, depends periodically on time due to the periodic driving; this allows the Fourier expansion

$$\Delta = \Delta_0 + \sum_{s=1}^{\infty} \Delta_s\cos(s\Omega t + \alpha_s), \tag{5}$$

with appropriate phases α_s, where Δ_0 is the time-independent (dc) component and Δ_s is the time-dependent (ac) one due to the periodic forcing.

A convenient way to deal with a set of Langevin equations is to introduce an appropriate probability density and to resort to the associated Fokker-Planck equation [21]. In general, the set of N Langevin equations (1) makes it necessary to consider the N-oscillator probability density $P(\{\phi_i\},t)$ and the corresponding Fokker-Planck equation [9]. In the system with global coupling, however, the set in Eq. (1) naturally reduces to the single Langevin equation (4), as shown above. This in turn leads to the Fokker-Planck equation for the single-oscillator probability density $P(\phi,t)$ with the self-consistency for the order parameter explicitly imposed, which has been considered in the absence of driving [8].

The Fokker-Planck equation for the probability density $P(\phi,t)$ reads [21]

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial\phi}\left[\left(\frac{\partial V(\phi)}{\partial\phi} - I\cos\Omega t\right)P\right] + D\frac{\partial^2 P}{\partial\phi^2}, \tag{6}$$

where $V(\phi)\equiv -K\Delta\cos\phi - \omega\phi$ is the washboard potential. Unlike the system without driving ($I=0$), the stationary solution of which has been obtained [8], Eq. (6) does not allow such a simple stationary solution. We thus use the periodicity of the system and expand the probability density as a Fourier series

$$P(\phi,t) = \sum_{n=-\infty}^{\infty} C_n(t)e^{in\phi}, \tag{7}$$

which, upon substitution into Eq. (6), yields

$$\dot{C}_n(t) = -[in(\omega + I\cos\Omega t) + n^2 D]C_n(t) - \frac{n}{2}K\Delta(t)C_{n+1}(t) + \frac{n}{2}K\Delta(t)C_{n-1}(t). \tag{8}$$

Since the probability density should be real, we have the relation $C_n = C^*_{-n}$; the normalization condition $\int_0^{2\pi}P(\phi,t)d\phi = 1$ gives the constant term $C_0 = 1/2\pi$. The differential recurrence relation in Eq. (8) can be written in the form of an integral recurrence equation

$$\begin{aligned} C_n(t) = {} & C_n(0)\exp\left[-in\left(\omega t + \frac{I}{\Omega}\sin\Omega t\right) - n^2 Dt\right] \\ & - \frac{n}{2}K\exp\left[-in\left(\omega t + \frac{I}{\Omega}\sin\Omega t\right) - n^2 Dt\right] \\ & \times \int_0^t dt'\,\Delta(t')[C_{n+1}(t') - C_{n-1}(t')] \\ & \times \exp\left[in\left(\omega t' + \frac{I}{\Omega}\sin\Omega t'\right) + n^2 Dt'\right], \end{aligned} \tag{9}$$

which is of the same form as the equation for a single oscillator [22], except for that, here, self-consistency for the order parameter is required.

III. SELF-CONSISTENCY EQUATION FOR THE ORDER PARAMETER

In this section we derive the self-consistency equation for the order parameter, which describes the response of the phase and determines the collective behavior of the system.

We suppose that the periodic driving amplitude I is distributed according to $f(I)$, independently of the natural frequency ω. Recalling that ϕ in Eq. (4) in fact represents $\phi - \theta$, we have the self-consistency equation

$$\Delta = \frac{1}{N}\sum_j e^{i\phi_j} = \int_{-\infty}^{\infty} dI f(I) \int_{-\infty}^{\infty} d\omega g(\omega) \langle e^{i\phi}\rangle_{\omega,I}, \tag{10}$$

where $\langle \cdots \rangle_{\omega,I}$ denotes the average with given ω and I.

With the probability density $P(\phi,t)$, the expansion of which is given by Eq. (7), we compute the average

$$\langle e^{i\phi}\rangle \equiv \int_0^{2\pi} d\phi \; e^{i\phi} P(\phi,t) = 2\pi C_1^*(t), \tag{11}$$

where the relation $C_n = C_{-n}^*$ has been used. Then Eq. (10) leads straightforwardly to

$$\Delta = 2\pi \int_{-\infty}^{\infty} dI f(I) \int_{-\infty}^{\infty} d\omega g(\omega) C_1^*(t), \tag{12}$$

which gives the order parameter in terms of the Fourier coefficient $C_1(t)$. Assuming $K\Delta \ll 1$ near the transition to the coherent state, we need to obtain C_1 up to the order of $(K\Delta)^3$. For this, we first compute C_2 from Eq. (9), neglecting C_3, and substitute the obtained C_2 back into the equation for C_1, i.e., Eq. (9) with $n=1$. At long times the transient terms such as $\exp[-n^2 Dt]$ for $n \neq 0$ vanish, and a lengthy calculation yields

$$\begin{aligned} C_1(t) = &-\frac{iK}{8\pi}\sum_{s=0}^{\infty} \Delta_s \sum_{\ell,m} J_\ell(x) J_m(x) \left[\frac{e^{i(\ell-m+s)\Omega t + i\alpha_s}}{\omega+(\ell+s)\Omega - iD} + \frac{e^{i(\ell-m-s)\Omega t - i\alpha_s}}{\omega+(\ell-s)\Omega - iD}\right] \\ &-\frac{iK^3}{64\pi}\sum_{s,s',s''} \Delta_s \Delta_{s'} \Delta_{s''} \sum_{\ell,\ell',m,m',n,n'} J_\ell(x) J_{\ell'}(x) J_m(2x) J_{m'}(2x) J_n(x) J_{n'}(x) \\ &\times [e^{i(\alpha_s-\alpha_{s'}-\alpha_{s''})}F(t;s,s',s'') + e^{i(\alpha_s+\alpha_{s'}-\alpha_{s''})}F(t;s,-s',s'') \\ &+ e^{i(\alpha_s-\alpha_{s'}+\alpha_{s''})}F(t;-s,s',s) + e^{i(\alpha_s+\alpha_{s'}+\alpha_{s''})}F(t;-s,-s',s) \\ &+ e^{i(-\alpha_s-\alpha_{s'}-\alpha_{s''})}F(t;s,s',-s) + e^{i(-\alpha_s+\alpha_{s'}-\alpha_{s''})}F(t;s,-s',-s) \\ &+ e^{i(-\alpha_s-\alpha_{s'}+\alpha_{s''})}F(t;-s,s',-s'') + e^{i(-\alpha_s-\alpha_{s'}-\alpha_{s''})}F(t;-s,-s',-s'')], \end{aligned} \tag{13}$$

where $x \equiv I/\Omega$, $\alpha_0 \equiv 0$, and the function $F(t;s,s',s'')$ depends on the indices ℓ, ℓ', m, m', n, and n' as well as on ω and Ω:

$$F(t;s,s',s'') \equiv \frac{e^{i(\ell-\ell'+m-m'+n-n'-s-s'-s'')\Omega t}}{[\omega+(n'+s)\Omega - iD][2\omega+(m'-n+n'+s+s')\Omega - 4iD]} \times \frac{1}{\omega+(\ell'-m+m'-n+n'+s+s'+s'')\Omega - iD}.$$

With the above expression for C_1, Eq. (12) yields the explicit form of the self-consistency equation for the order parameter.

Comparing term by term in the resulting self-consistency equation, we can determine each component of the order parameter. Namely, the dc component Δ_0 is given by the constant (zero-frequency) terms in the expansion of $C_1(t)$. The next component Δ_1 can be obtained from the terms with frequency Ω, the component Δ_2 from the 2Ω terms, and so on. For weak driving, the ac components of the order parameter are much smaller than the dc component, leading to the simple self-consistency equation:

$$\Delta \approx aK\Delta_0 - b(K\Delta_0)^3 \tag{14}$$

with the coefficients

$$\begin{aligned} a &= \frac{i}{2}\sum_{\ell,m} \int dI f(I) J_\ell(x) J_m(x) e^{i(\ell-m)\Omega t} \times \int d\omega \frac{g(\omega)}{\omega+m\Omega+iD}, \\ b &= -\frac{i}{4}\sum_{\ell,\ell',m,m',n,n'} \int dI f(I) J_\ell(x) J_{\ell'}(x) J_m(2x) \times J_{m'}(2x) J_n(x) J_{n'}(x) e^{i(\ell'-\ell+m'-m+n'-n)\Omega t} \\ &\times \int d\omega \frac{g(\omega)}{[\omega+n'\Omega+iD][2\omega+(m'+n'-n)\Omega+4iD]} \times \frac{1}{\omega+(\ell'+m'-m+n'-n)\Omega+iD}. \end{aligned} \tag{15}$$

In the simple case of no external driving ($I=0$) and noise ($D\to 0$), the representation $\pi\delta(\omega)=D(\omega^2+D^2)^{-1}$ in the limit $D\to 0$, together with the symmetry of $g(\omega)$, reduces Eq. (15) to $a=(\pi/2)g(0)$ and $b=-(\pi/16)g''(0)$, which indeed reproduces the self-consistency equation obtained in Ref. [5].

Solving Eq. (14), we obtain the collective behavior of the system, which has been analyzed in Ref. [10]; for small K, only the trivial solution $\Delta=0$ exists. On the other hand, for $K\geqslant K_c\equiv 1/a_0$, Eq. (14) also allows the nontrivial solution with the dc component

$$\Delta_0=\Delta_+\equiv\frac{\sqrt{b_0K(a_0K-1)}}{b_0K^2}, \tag{16}$$

where the constant coefficients are given by

$$a_0=\frac{i}{2}\sum_{\ell}\int dI f(I)J_{\ell}^2(x)\int d\omega\frac{g(\omega)}{\omega+\ell\Omega+iD},$$

$$b_0=-\frac{i}{4}\sum_{\ell,m,m',n,n'}\int dI f(I)J_{\ell}(x)J_m(2x)J_{m'}(2x)J_n(x)J_{n'}(x)J_{\ell+m+n-m'-n'}(x)$$
$$\times\int d\omega\frac{g(\omega)}{[\omega+n'\Omega+iD][\omega+\ell\Omega+iD][2\omega+(m'+n'-n)\Omega+4iD]}. \tag{17}$$

Thus as K is increased beyond K_c, the null solution becomes unstable and the stable nontrivial solution Δ_+ (accompanied by the ac components) appears via a pitchfork bifurcation at $K=K_c$ [10].

IV. NOISE EFFECTS ON SYNCHRONIZATION

To understand the cooperative effects of the driving and noise on the response of the system, we examine in this section how the noise affects synchronization behavior of the system. For simplicity, we consider the weak-driving or high-frequency limit ($x\equiv I/\Omega\ll 1$), expand the Bessel functions in Eq. (15) to the order of x^2, and perform the average over the distribution $f(I)$. This gives the coefficient a to the order of σ_I, the variance of the distribution $f(I)$:

$$a=\frac{D}{2}\int d\omega\frac{g(\omega)}{\omega^2+D^2}+\frac{\sigma_I}{8\Omega^2}$$
$$\times\int d\omega g(\omega)\Bigg[\frac{D(\cos 2\Omega t-2)}{\omega^2+D^2}$$
$$-\frac{D(\cos 2\Omega t-1)+(\omega+\Omega)\sin 2\Omega t}{(\omega+\Omega)^2+D^2}$$
$$-\frac{D(\cos 2\Omega t-1)-(\omega-\Omega)\sin 2\Omega t}{(\omega-\Omega)^2+D^2}$$
$$+\frac{D\cos 2\Omega t+(\omega+2\Omega)\sin 2\Omega t}{2(\omega+2\Omega)^2+2D^2}$$
$$+\frac{D\cos 2\Omega t-(\omega-2\Omega)\sin 2\Omega t}{2(\omega-2\Omega)^2+2D^2}\Bigg]$$
$$\equiv a_0+a_2\cos(2\Omega t+\alpha_2). \tag{18}$$

Similarly, a tedious but straightforward calculation leads to the coefficient $b=b_0+b_2\cos(2\Omega t+\alpha_2)$. Note that the symmetry of the distribution $f(I)$ about $I=0$ forbids the frequency Ω term, which is linear in the driving. Here it is easy to observe that a_0, which reads

$$a_0=\frac{D}{2}\left(1-\frac{\sigma_I}{2\Omega^2}\right)\int d\omega\frac{g(\omega)}{\omega^2+D^2}+\frac{D\sigma_I}{8\Omega^2}\int d\omega g(\omega)$$
$$\times\left[\frac{1}{(\omega+\Omega)^2+D^2}+\frac{1}{(\omega-\Omega)^2+D^2}\right], \tag{19}$$

in general, decreases monotonically with D. Thus the critical coupling strength K_c grows as the noise level is raised. Figure 1 displays the monotonic increase of $K_c(=a_0^{-1})$ with the noise level D, for $\sigma_I=0.1$ and $\Omega=2.0$. For the distribution of natural frequencies, the Gaussian distribution with variance $\sigma_\omega=0.5$ has been chosen.

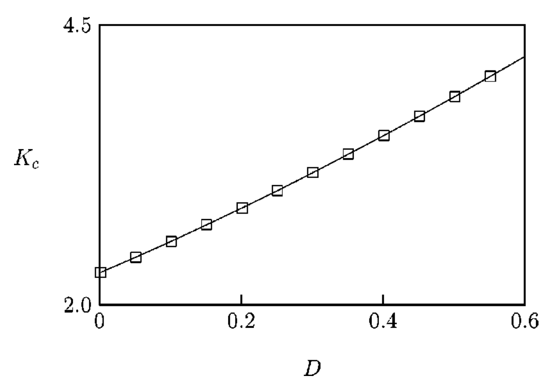

FIG. 1. Critical coupling strength beyond which synchronization appears versus the noise level in the system with the driving frequency $\Omega=2$ and the variances $\sigma_\omega=0.5$ and $\sigma_I=0.1$. The random noise in the system increases monotonically the critical coupling strength, thus tending to suppress synchronization.

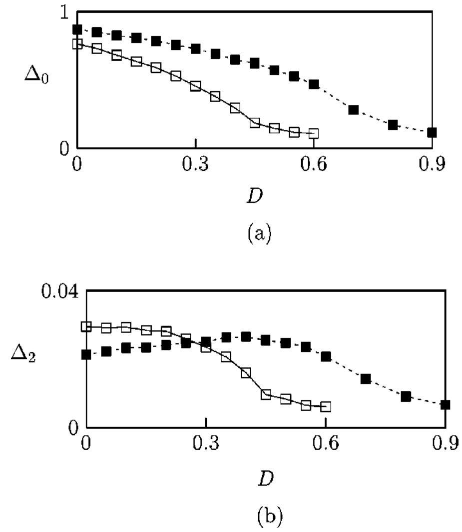

FIG. 2. Behavior of the order parameter in the presence of noise. The data represented by empty and solid squares correspond to the coupling strength $K=2.5$ and $K=3.0$, respectively. (a) The dc component Δ_0 for both $K=2.5$ and $K=3.0$ is shown to decrease monotonically. (b) The ac component Δ_2 for $K=2.5$ decreases monotonically, while for $K=3.0$ it displays a peak at a finite noise level. The standard deviations of the data (not shown) range from 5 to 15 %, and lines are merely guides to the eye.

With the coefficients a and b obtained above, the order parameter can be obtained from Eq. (14) and its behavior in the presence of noise can be investigated. Indeed the dc component Δ_0 given by Eq. (16) is easily found to decrease monotonically as the noise level D is increased. On the other hand, it is too complicated to obtain analytically the explicit behavior of the ac component Δ_s $(s\geqslant 1)$. Further, the analytical results are based on Eq. (14), which is valid only near the transition $(K\approx K_c)$; this makes it desirable to obtain the order parameter numerically. We have thus performed numerical simulations to compute the components Δ_0 and Δ_2. Since the effects of external driving first appear in the coefficients a_2 and b_2, giving rise to the frequency 2Ω term, it is relevant to investigate Δ_2 as the response to the external driving. In the simulations, Eq. (1) has been integrated with discrete time steps of $\Delta t=0.01$. At each run, we have used $N_t=6048$ time steps to compute the order parameter, discarding the data from the first 4×10^3 steps, and varied both Δt and N_t to verify that the stationary state was achieved. Finally, independent runs with 30 different distributions of the natural frequency and initial conditions have been performed, over which the averages have been taken. For both the distribution of the driving amplitudes and that of the natural frequencies, we have chosen Gaussian distributions with various values of variances σ_I and σ_ω, only to find no qualitative difference.

Figure 2 displays the obtained behaviors (a) of the dc component Δ_0 and (b) of the ac component Δ_2 in the system of $N=1000$ oscillators, driven by frequency $\Omega=1.0738$ and with variances $\sigma_\omega=1.0$ and $\sigma_I=1.0$. The data represented by empty and solid squares in Fig. 2 correspond to the coupling strength $K=2.5$ and $K=3.0$, respectively. It is shown that for both values of the coupling strength, Δ_0 decreases monotonically as the noise level D is raised. Such monotonic behavior is also exhibited by Δ_2 for $K=2.5$. For $K=3.0$, on the other hand, Fig. 2(b) displays that the ac component Δ_2 first increases with noise and reaches its maximum at a finite value of the noise level D. Similar nonmonotonic behavior of Δ_2 can be observed for larger values of the coupling strength K, suggesting the presence of SR-like behavior in the order parameter. Note, however, that the dc component Δ_0 is, in general, dominant over the ac component, leading to the monotonic decrease of the total order parameter. It is thus concluded that noise tends to suppress monotonically the overall synchronization in the system.

V. RESPONSE OF THE PHASE VELOCITY

In this section we investigate the power spectrum of the phase velocity at the driving frequency, which conveniently describes the response of the phase velocity to the external driving. In the case of a superconducting wire network or array, the phase velocity can be identified with the voltage via the Josephson relation, and the power spectrum of the phase velocity simply corresponds to the voltage power spectrum under the combined direct and alternating current driving. Equations (4) and (7) give the average phase velocity of a single oscillator in terms of $\mathrm{Im}\, C_1$, the imaginary part of C_1:

$$\langle\dot{\phi}\rangle \equiv \int_0^{2\pi} d\phi P(\phi,t)\dot{\phi}$$

$$= \omega + I\cos\Omega t + 2\pi K\Delta\, \mathrm{Im}\, C_1, \qquad (20)$$

which, upon substitution of Eq. (13) for $\mathrm{Im}\, C_1$, obtains the simple form

$$\langle\dot{\phi}\rangle = \omega + A\cos\Omega t + B\sin\Omega t + O(K^2\Delta_0\Delta_1), \qquad (21)$$

with the amplitudes

$$A = I\left[1 - \frac{K^2\Delta_0^2}{2\Omega}\left(\frac{\omega+\Omega}{(\omega+\Omega)^2+D^2} - \frac{\omega-\Omega}{(\omega-\Omega)^2+D^2}\right)\right],$$

$$B = \frac{K^2\Delta_0^2}{2\Omega} DI\left[\frac{1}{(\omega+\Omega)^2+D^2} - \frac{2}{\omega^2+D^2} + \frac{1}{(\omega-\Omega)^2+D^2}\right].$$

The desired power spectrum S of the phase velocity at the driving frequency is proportional to the square of the Fourier component of frequency Ω, i.e., $S(\Omega)\propto A^2+B^2$. In the limit $D\to 0$, the amplitude of the Fourier component approaches

$$I^2\left[1 + \frac{K^2\Delta_0^2}{\omega^2-\Omega^2}\right]^2$$

$$+ I^2\frac{\pi^2 K^4\Delta_0^4}{4\Omega^2}[\delta(\omega+\Omega) - 2\delta(\omega) + \delta(\omega-\Omega)]^2,$$

while it approaches I^2 in the limit $D\to\infty$. It is of interest to note that the amplitude in the noiseless limit can be either larger or smaller than that in the strong-noise limit, depend-

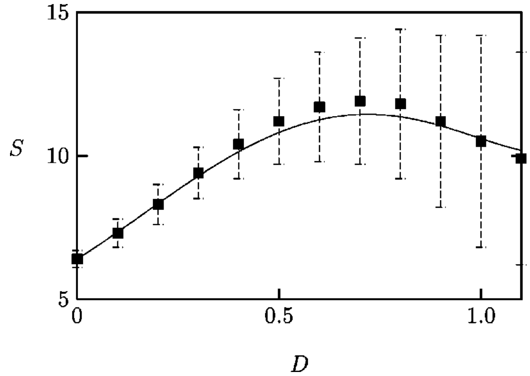

FIG. 3. Noise subtracted power spectrum of the phase velocity at the driving frequency. There appears an optimal noise level at which the power spectrum reaches its maximum. The error bars have been estimated by the standard deviation and the line is merely a guide to the eye.

ing on ω and Ω; as the noise level is raised, the amplitude tends to decrease from the noiseless value for $\omega > \Omega$ and increase for $\omega < \Omega$. Accordingly, for given driving frequency, those oscillators with smaller/larger natural frequencies contribute to the increase/decrease of the amplitude toward its asymptotic value I^2. For small values of the variance σ_ω, for example, most oscillators should possess the natural frequency $\omega < \Omega$, although there may still exist some oscillators with frequency $\omega > \Omega$. The power spectrum of the whole system, which is given by the sum of contributions from all the oscillators, is then expected to increase for small D and to approach the asymptotic value which is proportional to I^2. Unfortunately, however, the approximations used on various stages of the analysis disallow a reliable analysis. In particular, the extrapolation to the limit $D \to 0$ is untrustworthy since nonzero effective temperature $(D \neq 0)$ has been assumed in solving the equation for $P(\phi,t)$. It is also obvious that the higher-order terms neglected in the analysis set a limit in the regime of validity, making it desirable to investigate the system by other means.

We have thus performed numerical simulations to obtain the power spectrum for various values of the coupling strength and of the variance in the distributions of the natural frequency and of the driving amplitude. We have again integrated Eq. (1) for the system of $N = 1000$ oscillators with discrete time steps of $\Delta t = 0.01$, using at each run $N_t = 6048$ time steps to compute the power spectrum of the phase velocity and discarding the data from the first 4×10^3 steps. The averages have been taken over 300 independent runs with different distributions of the natural frequency and initial conditions. From the obtained time series, we have computed the power spectrum by means of the fast Fourier transform algorithm. To take into account the background noise, we have taken five nearest data points around the peak at the driving frequency in the power spectrum and performed the average to give the noise level. (The results have been found not to change qualitatively even if other measure for the noise level is adopted.)

In Fig. 3 we present the obtained data: the background noise subtracted power spectrum of the phase velocity at the driving frequency versus the noise level. For the distributions of the natural frequency and of the driving amplitude, Gaussian ones have been chosen with variances $\sigma_\omega = 0.5$ and $\sigma_I = 0.2$, respectively, while the coupling strength $K = 2.5$ and the driving frequency $\Omega = \pi/1.024$ have been taken. Remarkably, Fig. 3 displays that the power spectrum increases as the noise level is raised from zero. Obviously, it does not keep increasing monotonically with the noise, and there apparently exists an optimal noise level at which the power spectrum reaches its maximum. Beyond the optimal noise level, the power spectrum first falls off gradually and saturates eventually toward its asymptotic value, although this behavior is somewhat obscured by the large fluctuations due to strong noise. Such a broad peak followed by gradual decrease has also been observed in the SR of another system [15]. It is thus suggested that the response of the phase velocity also displays SR-like behavior in the appropriate regime. Since the phase velocity corresponds to the voltage in a superconducting system, this indicates that the noise subtracted power spectrum of the voltage displays such resonance behavior.

VI. SUMMARY

We have studied the noise effects in a driven system of globally coupled oscillators, with emphasis on the interplay of noise and periodic driving. In particular, to investigate the possibility of resonance behavior, we have considered the response of the phase velocity, as well as the order parameter which describes the phase synchronization. The self-consistency equation for the order parameter, derived from the recurrence relation of the probability density, has been shown to display monotonic decrease of the total order parameter in the presence of noise. It has thus been concluded that noise, in general, suppresses overall phase synchronization in the system, i.e., superconductivity tends to be disturbed by noise present in a superconducting wire network or array.

Nevertheless, it has also been revealed that for large coupling strengths the ac component of the order parameter increases with the noise level growing from zero and reaches its maximum at a finite noise level. Such resonance behavior has also been observed in the response of the phase velocity; at low noise levels, the noise subtracted power spectrum of the phase velocity has been found to increase with noise, displaying a broad peak at a finite noise level. As the noise level is raised further, the power spectrum appears to saturate toward its asymptotic value, although concealed by large fluctuations due to strong noise. In conclusion, the phase synchronization, describing the collective behavior of the coupled-oscillator system, is suppressed monotonically in the presence of noise. Still, the responses of the phase and of the phase velocity can display nonmonotonic resonance behavior in the appropriate regime, which may be manifested by a broad resonance peak of the voltage power spectrum in the case of a superconducting system.

It is also of interest to note that the phase velocity on average may serve as a measure of $\dot{\Delta}/\Delta$, the rate of change of phase synchronization. Accordingly, the resonance behavior in the response of the phase velocity suggests that the approach to the coherent state with synchronization can be accelerated by the presence of weak noise. Since the phase synchronization corresponds to the memory retrieval in the network of neuronal oscillators [8,9], the resonance behavior may also imply the information processing assisted by weak noise in a biological system, e.g., the crayfish who appears to use such resonance to perceive an enemy quickly. The detailed understanding of the resonance behavior and its implications to applicable physical and biological systems require more extensive analytical and numerical investigations, which are left for further study.

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